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by
MICHAEL L. PINEDO
and
SHELDON M. ROSS

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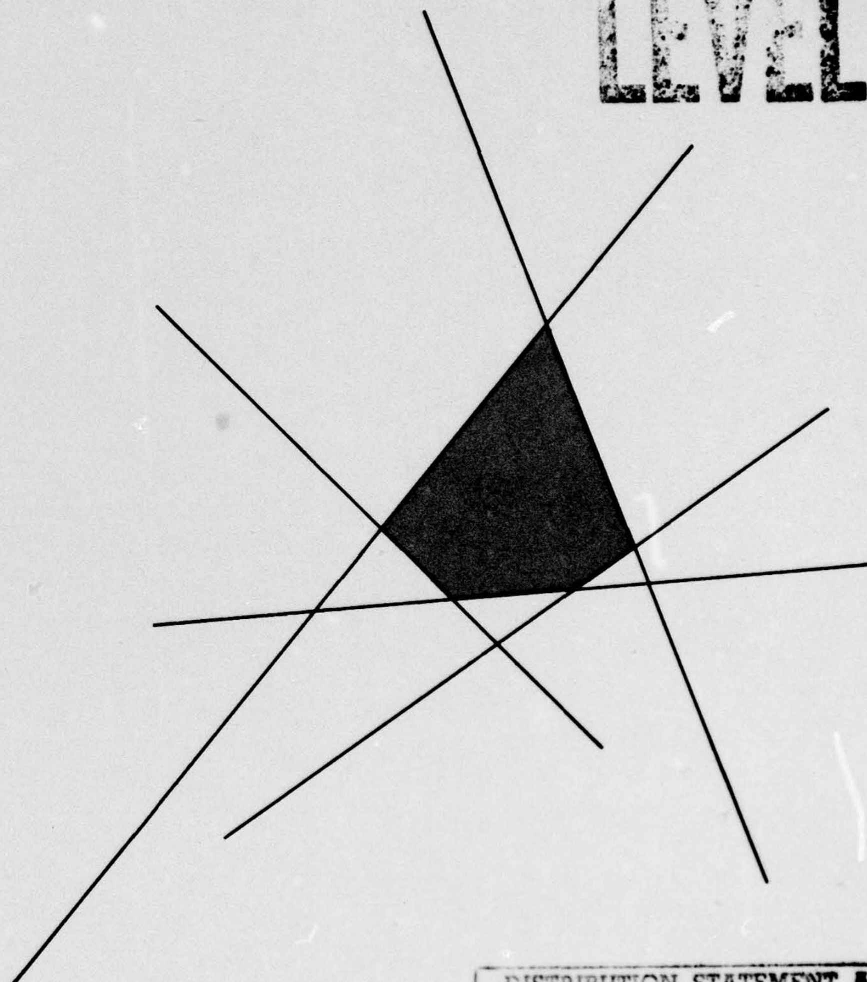

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SCHEDULING JOBS SUBJECT TO NONHOMOGENEOUS POISSON SHOCKS

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ABSTRACT

Consider n jobs which have to be performed sequentially in time. There are external shocks which occur according to a nonhomogenous Poisson process. If a shock occurs during the performance of a job, then work on that job ends and work on the next one commences. A job is successfully performed if no shocks occur during its execution time. We consider such problems as maximizing:

~~(i)~~(1) The expected number of successful job performances;
~~(ii)~~(2) The length of time until no jobs remain; and
~~(iii)~~(3) The expected total reward earned; where a reward R_i is obtained upon successful completion of job i . R sub i

We determine conditions on the distribution of job performances which result in simple policies being optimal.

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SCHEDULING JOBS SUBJECT TO NONHOMOGENEOUS POISSON SHOCKS

by

Michael L. Pinedo and Sheldon M. Ross

1. INTRODUCTION AND SUMMARY

Suppose that n jobs have to be performed sequentially in time - the i^{th} job requiring a random time X_i for its execution. In addition, suppose there are external shocks which occur according to a non-homogeneous Poisson process. If a shock occurs during the performance of a job then work on that job ends and work on the next one commences. A job is said to be successfully performed if no shocks occur during its execution time.

We are interested in determining the job schedule that maximizes the expected number of successful job performances. However, as a means to determining this, we shall first consider the related problem of stochastically maximizing the length of time until all jobs are finished (either successfully or by shocks). This related problem is of independent interest for, by interpreting the n jobs as being n spares in a stockpile, it becomes one of stochastically maximizing the life of a stockpile of spares which are subject to shocks which kill any spare in use when the shock occurs.

The related problem, without any assumption of external shocks, but under the condition that the lifetime of a spare depends on its time of installation has previously been studied in [1] and [2]. Brown and Solomon [2] established optimal schedules for certain cases in which the lifetimes of the spares are monotone likelihood ratio ordered.

In this paper we consider a more general ordering than monotone likelihood ratio ordering. This ordering, which we call failure rate ordering, implies stochastic ordering. We show in Section 3 that if the lifetime distributions are failure rate ordered and if the intensity function of the Poisson process is increasing, then in the related stockpile problem, the strategy of issuing spares is decreasing order of their means stochastically maximizes the system life and in the original problem, the strategy of issuing jobs in increasing order of their means maximizes the expected number of successful jobs. Also, when the intensity function of shocks is decreasing, the optimal strategy is to reverse the order given above.

In Section 4 we suppose that the execution time of job i is exponentially distributed with rate λ_i and that a reward R_i is earned if job i is successfully performed, $i = 1, \ldots, n$. We show that if the intensity function of shocks is increasing then $1, 2, \ldots, n$ is the optimal strategy if both R_i and $\lambda_i R_i$ are decreasing in i. If the intensity function of shocks is decreasing then $n, n-1, \ldots, 1$ is the optimal strategy if both R_i and $\lambda_i R_i$ are decreasing and λ_i is increasing in i.

2. DEFINITIONS AND PRELIMINARY RESULTS

Two random variables X_1 and X_2 are said to be increasing failure rate ordered if

$$\bar{F}_1(t)/\bar{F}_1(s) \geq \bar{F}_2(t)/\bar{F}_2(s) \quad \text{for all} \quad s < t$$

where $\bar{F}_i(t) = P\{X_i > t\}$. Given a collection of pairwise failure rate ordered random variables we can reorder them such that X_i and X_j are increasing failure rate ordered if $i < j$. We call such a collection increasing failure rate ordered.

If the X_i are continuous with densities f_i then the failure rate function r_i of X_i is defined by $r_i(t) = f_i(t)/\bar{F}_i(t)$. The following lemma justifies our terminology.

Lemma 1:

The continuous random variables X_1 and X_2 are increasing failure rate ordered if and only if $r_1(t) \leq r_2(t)$ for all t .

Proof:

Using the well known identity

$$\bar{F}(t) = \exp\left[-\int_0^t r(y)dy\right]$$

we have that

$$\bar{F}_i(t)/\bar{F}_i(s) = \exp\left[-\int_s^t r_i(y)dy\right]$$

and so if X_1 and X_2 are increasing failure rate ordered then

$$\int_s^t r_1(y)dy \leq \int_s^t r_2(y)dy \quad \text{for all} \quad s , t$$

implying that $r_1(y) \leq r_2(y)$. The converse also follows by reversing the argument.||

Two continuous random variables X_1 and X_2 are said to be increasing monotone likelihood ratio ordered if for $s < t$

$$f_2(t)/f_1(t) \geq f_2(s)/f_1(s) .$$

From the definition of monotone likelihood ratio ordering we have

$$\int_s^\infty (f_2(t)f_1(s) - f_2(s)f_1(t))dt \geq 0 \quad \text{for all} \quad s$$

or, equivalently

$$r_1(s) \geq r_2(s) .$$

As it easily follows that failure rate ordering implies stochastic ordering from the identity $\bar{F}(t) = \exp\left\{- \int_0^t r(s)ds\right\}$ we see that monotone likelihood ratio ordered $\Rightarrow$ failure rate ordered $\Rightarrow$ stochastic ordered.

3. RESULTS

We consider a stockpile of n spares in which the i^{th} spare has lifetime X_i . We have a one component system subject to shocks which occur according to a nonhomogenous Poisson process with intensity function $v(t)$. A spare has to be replaced when it fails as the result of its internal lifetime or as the result of a shock.

Theorem 1:

Given n spares with lifetimes $X_1, \ldots, X_n$ which are increasing failure rate ordered and given that the rate $v(t)$ of the shocks is increasing (decreasing) over time, then the schedule $1,2, \ldots, n$ $(n, n-1, \ldots, 1)$ stochastically maximizes the lifetime of the system.

Proof:

We will prove only the case where $v(t)$ is increasing over time, as the proof for $v(t)$ decreasing is identical.

First we consider the case $n = 2$. To show the probability that system life will last until time t is larger using the alleged optimal schedule, we condition on the number of shocks that occur before time t . If there are zero shocks, both schedules will yield the same probability. If there are two or more shocks before time t , the probability will be zero under both schedules. So it suffices to consider the case in which only one shock occurs before time t , because only then will there be a difference between the two schedules. If $P_1(x)$ denotes the probability that this shock occurs before time x $(x < t)$, then $P_1(x) = \int_0^x v(s)ds / \int_0^t v(s)ds$. Letting α_1 (α_2) denote the conditional probability that the system will function until time t , given only one shock occurs and using schedule $1,2$ $(2,1)$, then

$$\alpha_1 = \int_0^t \bar{F}_1(x)\bar{F}_2(t - x)v(x)dx / \int_0^t v(s)ds$$

and

$$\alpha_2 = \int_0^t \bar{F}_2(x)\bar{F}_1(t - x)v(s)ds / \int_0^t v(s)ds \ .$$

Hence we must show

$$\int_0^{t/2} [\bar{F}_1(x)\bar{F}_2(t - x) - \bar{F}_2(x)\bar{F}_1(t - x)]v(x)dx$$

$$\geq \int_{t/2}^{t} [\bar{F}_2(x)\bar{F}_1(t - x) - \bar{F}_1(x)\bar{F}_2(t - x)]v(x)dx$$

or, equivalently,

$$\int_{t/2}^{t} [\bar{F}_2(x)\bar{F}_1(t - x) - \bar{F}_2(t - x)\bar{F}_1(x)][v(t - x) - v(x)]dx \geq 0 \ .$$

Now in the range of integration $v(x) \leq v(t-x)$, so the inequality holds if

$$\bar{F}_1(x)\bar{F}_2(t - x) \leq \bar{F}_2(x)\bar{F}_1(t - x)$$

which follows since X_1 and X_2 are increasing failure rate ordered. This completes the proof for the case $n = 2$.

We will now show that system life under arbitrary schedule $1',2',3', \ldots, m'$ is stochastically larger than system life under schedule $2',1',3', \ldots, m'$ if $E(X_{1'}) > E(X_{2'})$. Let t_i be the time epoch of shock i . Condition on t_i , $i = 1,2,3, \ldots$, and on the lifetimes of the spares $3',4', \ldots, m'$. Now let $G(s)$ be the conditional

lifetime of the system given that the third spare is installed at time epoch s and the subsequent spares are installed according to the schedule $3',4', \ldots, m'$. We will show that $G(s) \uparrow s$. We first note that the third spare (with lifetime $X_{3'}$) will be installed at time epoch s. Suppose the next shock occurs at time epoch w. Then the next spare will be installed at time epoch $s + \min(X_{3'}, w - s)$. This last expression is nondecreasing in s. Repeating this argument proves $G(s) \uparrow s$. From the case $n = 2$, we know the time the third spare will be installed is stochastically larger under schedule $1',2'$. As stochastic ordering is preserved under monotone transformations the conditional system life under schedule $1',2',3', \ldots, m'$ is stochastically larger than under schedule $2',1',3', \ldots, m'$. Unconditioning gives the desired result.

From this we easily can obtain a more general result: the system life under arbitrary schedule $1',2', \ldots, i',j',k', \ldots, n'$ is stochastically larger than under schedule $1',2', \ldots, i',k',j', \ldots, n'$ if $E(X_{j'}) > E(X_{k'})$. This can be shown by conditioning on the time epoch spare i' dies and using the preceding results.

As any schedule can be transformed through pairwise switches into schedule $1,2, \ldots, n$ in such a way that each pairwise switch gives an improvement in the lifetime of the system, it is clear that schedule $1,2, \ldots, n$ is the optimal schedule.||

In exactly the same way, we can also prove that the reverse schedule to that given in Theorem 1 stochastically minimizes the lifetime of the system. That is, we have

Corollary 1:

Under the conditions of Theorem 1, if $v(t)$ is increasing (decreasing) then the schedule $n, n-1, \ldots, 2, 1$ $(1, 2, \ldots, n-1, n)$ stochastically minimizes the lifetime of the system.

We are now in position to prove

Theorem 2:

Given n spares with lifetimes $X_1, \ldots, X_n$ which are increasing failure rate ordered and given that the intensity rate $v(t)$ of shocks is increasing (decreasing) over time, then the schedule $n, n-1, \ldots, 1$ $(1, 2, \ldots, n)$ maximizes the expected number of successful job performances.

Proof:

Let $N(t)$ denote the number of shocks by time t and for any schedule π let L_π denote the system life under policy π. Now $\left\{N(t) - \int_0^t v(s)ds\right\}$ is a Martingale with 0 mean and as L_π is a stopping time having a finite expected value, it follows from Martingale theory that, for any schedule π,

$$E\left[N(L_\pi) - \int_0^{L_\pi} v(s)ds\right] = 0$$

or

$$E[N(L_\pi)] = E\left[\int_0^{L_\pi} v(s)ds\right].$$

Now letting π^* denote the allegedly optimal policy it follows from Corollary 2 that L_{π^*} is stochastically smaller than L_π and as $\int_0^x v(s)ds$ is a monotone function of x it follows that

$$E\left[\int_0^{L_{\pi^*}} v(s)ds\right] \leq E\left[\int_0^{L_\pi} v(s)ds\right]$$

implying that

$$E[N(L_{\pi^*}) \leq E[(L_\pi)] .$$

The result now follows since $N(L_\pi)$ is just equal to n minus the number of successful job performances.||

<u>Remarks</u>:

1. It may be assumed that each time a spare (or job) is installed the decision-maker knows the lives of the previous installed spares (or jobs). Conceivably he may take this into account at each stage in deciding which spare to install next. We claim that our schedules remain optimal even among the larger class of policies which allow the decision-maker to operate sequentially. To see this we argue as follows: If there is one spare left, there is no decision to be made. When there are two spares left, our claim is obviously true. When there are three spares left, we know that whichever spare is installed, independent of its lifetime, the last two spares will be installed according to our optimal policy. The objective is, to choose the spare which

combined with the last two spares (ordered according to our optimal schedule) gives the largest total lifetime. So if we operate sequentially, our decision with three spares left will result in the same schedule as when we do not operate sequentially. Repeating this argument for a larger number of spares completes the proof.

2. If the distributions of the lifetimes of the spares are merely stochastically ordered, then the schedules described in Therems 1 and 2 are not necessarily optimal. We give the following counterexample. Let

$$v(x) = \begin{cases} 10^{-6} & \text{for} \quad 0 < x < 2 \\ 0 & \qquad\quad x > 2 \end{cases}$$

and suppose we have 2 spares with lifetimes X_1 and X_2 which are such that

$$X_1 = \begin{cases} 1 & \text{with probability} \quad 0.5 \\ 3 & \text{with probability} \quad 0.5 \end{cases}$$

$$X_2 = \begin{cases} 2 & \text{with probability} \quad 0.5 \\ 3 & \text{with probability} \quad 0.5 \,. \end{cases}$$

Obviously X_2 is stochastically larger than X_1, but X_1 and X_2 are not failure rate ordered. The schedule which maximizes the expected lifetime of the system is 2,1 . This can be easily checked by considering the case of only one shock occurring in [0,2] , which due to the low rate of shocks is the dominant probability.

3. One possibility we have not allowed for is for a still functioning spare to be replaced and kept in the stockpile for later use. Let us assume in this case that the remaining life of the spare is not changed by such a preemption. Then our results would go through only under additional conditions. Specifically to prove that the given schedule in Theorem 1 remains optimal and that a preemption is never optimal we would have to assume that the component life distributions all have decreasing failure rates in the case where $v(t)$ is increasing and all have increasing failure rates when $v(t)$ is decreasing. Similarly in Theorem 2 we would need that the spares all had increasing (decreasing) failure rates in the case where $v(t)$ increases (decreases). The proof follows from noting that under the presumed ordering, at any time a preemption is contemplated the remaining lifetime of the spare to be preempted is still ordered among the remaining spares in the same way as originally. The above can be made the first step of an inductive proof which proves the result when at most k preemptions are allowed.
4. Since many individual spares are built up of smaller parts it may be worthwile to note that a k-of-n system in which each component life is independent and has the same life distribution F would have a failure rate order relationship to a similar k-of-n having component life distributions all equal to G, whenever F and G have that failure rate ordered relationship.
5. Both problems could also be analyzed using the approach of Brown and Solomon [2]. Using their lemma 1 on the case $n = 2$ results in a very simple proof. However, this proof is only valid for monotone likelihood ratio ordered and not failure rate ordered distributions.

4. EXPONENTIAL LIFETIMES AND REWARDS

In this section we suppose that X_i, the execution time of job i, is exponentially distributed with rate λ_i, $i = 1, \ldots, n$. In addition, we suppose that a reward R_i is earned if job i is successfully performed.

Denote by T_i the length of time that work is done on job i, i.e., it is the length of time from the beginning of work on job i until work on that job ends either because it is completed successfully or a shock occurs. We then have the following result from Derman-Lieberman-Ross (Page 559 of [3]).

Lemma 2:

For any scheduling policy π

$$E_\pi[\text{Total Reward}] = \sum_{i=1}^{n} \lambda_i R_i E_\pi[T_i] .$$

In words, Lemma 2 says that the expected return is the same if one received rewards at a constant rate $\lambda_i R_i$ whenever job i was being performed.

Lemma 3:

If $v(t)$ is increasing and if $\lambda_i < \lambda_j$ and $\lambda_i R_i = \max_k \lambda_k R_k$, then the total expected reward under schedule $i, j, k_1, \ldots, k_{n-2}$ is larger than the total expected reward under schedule $j, i, k_1, \ldots, k_{n-2}$ where $k_1, \ldots, k_{n-2}$ can be any permutation of the remaining $n - 2$ jobs.

Proof:

Schedule $i,j,k_1, \ldots, k_{n-2}$ will be denoted by π, and $j,i,k_1, \ldots, k_{n-2}$ by π'. According to Lemma 2, it suffices to show that

$$
\begin{aligned}
&\lambda_i R_i E_\pi[T_i] + \lambda_j R_j E_\pi[T_j] + \sum_{i=1}^{n-2} \lambda_{k_i} R_{k_i} E_\pi\left[T_{k_i}\right] \\
(1)\quad &\geq \lambda_i R_i E_{\pi'}[T_i] + \lambda_j R_j E_{\pi'}[T_j] + \sum_{i=1}^{n-2} \lambda_{k_i} R_{k_i} E_{\pi'}\left[T_{k_i}\right] .
\end{aligned}
$$

Now it follows from the proof of Theorem 1, that each of the jobs $j,k_1, \ldots, k_{n-2}$ will be started at a stochastically earlier time under π' than under π. Hence as $v(t)$ is increasing we have that

$$
\begin{aligned}
&E_\pi[T_j] \leq E_{\pi'}[T_j] \quad \text{and} \\
(2)\quad &E_\pi\left[T_{k_i}\right] \leq E_{\pi'}\left[T_{k_i}\right], \; i = 1, \ldots, n-2 .
\end{aligned}
$$

But from Theorem 1, we have that

$$
\begin{aligned}
&E_\pi[T_i] + E_\pi[T_j] + \sum_{k=1}^{n-2} E_\pi\left[T_{k_i}\right] \\
(3)\quad &\geq E_{\pi'}[T_i] + E_{\pi'}[T_j] + \sum_{k=1}^{n-2} E_{\pi'}\left[T_{k_i}\right] .
\end{aligned}
$$

Now (2) and (3) in conjunction with the condition $\lambda_i R_i \geq \lambda_k R_k$ for all k establishes (1).||

We now have

Theorem 3:

If $v(t)$ is increasing and if R_i and $\lambda_i R_i$ are decreasing, then $1, 2, \ldots, n$ is the optimal schedule.

Proof:

The proof is by contradiction. Consider any strategy which is not in decreasing order of R. Suppose the job with the highest value of R (spare 1) is not scheduled first. Suppose the job which immediately precedes job 1 is job k.

We consider two cases:

Case 1:

$R_1 > R_k$, $\lambda_1 < \lambda_k$ and $\lambda_1 R_1 > \lambda_k R_k$.

According to Lemma 3, interchanging job 1 and job k increases the total expected reward.

Case 2:

$R_1 > R_k$ and $\lambda_1 > \lambda_k$. Let $P_i(\pi)$ denote the probability that job i is successfully performed under the original schedule and let $P_i(\pi')$ denote the probability that job i is successfully performed under the schedule obtained after performing the pairwise switch between jobs k and 1. Now since $v(t)$ is increasing, we have that

$$P_1(\pi) \geq P_1(\pi')$$

and from Theorem 2, we have

$$P_1(\pi) + P_k(\pi) \geq P_1(\pi') + P_k(\pi') .$$

Since $R_1 \geq R_k$, it follows from the above that

$$P_1(\pi)R_1 + P_k(\pi)R_k \geq P_1(\pi')R_1 + P_k(\pi')R_k .$$

According to Theorem 1, all the jobs scheduled after jobs 1 and k start their execution stochastically earlier under schedule π' . Therefore, the probability of each of these jobs being successfully performed is larger under π' . Thus, interchanging job k and job 1 increases the total expected reward. So it is clear that in any case the original schedule could not be optimal.

If the schedule is not 1,2, ..., n we always will be able to find a job with a reward bigger than each of the jobs following it and the one preceding it. Performing a pairwise switch increases the total expected reward.||

One remark should be made here. It is not necessarily true that if v(t) is increasing, then the optimal schedule is to do jobs in decreasing value of R . To obtain a counterexample, suppose that $\lambda_i R_i = \lambda_j R_j$ for all i , j . Hence, from Lemma 2 and Theorem 1, it follows that the optimal policy is to schedule jobs in decreasing order of λ_i ; and therefore, all policies are not equivalent as would be the case if the conjecture were true.

We now consider the case v(t) decreasing.

<u>Theorem 4</u>:

If v(t) is decreasing and if R_i and $\lambda_i R_i$ are increasing and λ_i is decreasing, then 1,2, ..., n is the optimal schedule.

<u>Proof</u>:

The proof is similar to the proof of Case 2 in Theorem 3.||

Again, in this case, schedule 1,2, ..., n is not necessarily optimal under weaker sufficient conditions. Counterexamples can be found easily.

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